Geometrical review of the dome in Palau Güell

Revisión de la tipología geométrica de la cúpula del Palau Güell

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ABSTRACT

The Palau Güell, by Antoni Gaudí, is a UNESCO World Heritage Site. Architects and historians commonly claim that the surface of the dome in Palau Güell's Central Hall is a paraboloid. In this study, using photogrammetrical and geometrical techniques, we prove this claim to be wrong and we define the quadric surface which best fits the dome. Besides, we provide an objective measure of that fit and we state the geometric parameters defining this surface.

Keywords: Palau Güell, Antoni Gaudí, dome, paraboloid, ellipsoid.

RESUMEN

El Palau Güell de Antoni Gaudí está catalogado por la UNESCO como World Heritage. Es comúnmente afirmado por arquitectos e historiadores que la superficie del diseño de la cúpula que cubre el Salón Central del Palau Güell es un paraboloide. Mediante técnicas fotogramétricas y geométricas mostramos que tal afirmación no es cierta. Esta investigación determina cual es la superficie cuádrica que mejor ajusta a la cúpula. Además, damos una medida de ese ajuste, y mostramos cuáles son los parámetros geométricos que configuran esa superficie.

Palabras clave: Palau Güell, Antoni Gaudí, cúpula, paraboloide, elipsoide.

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1. INTRODUCTION

The Palau Güell (1885-1890) is amongst the first important projects by Antoni Gaudí (1852-1926). The building was commissioned by the Barcelona businessman Eusebi Güell Bacigalupi, who gave the architect total freedom in design. The dome of the Central Hall is one of the most characteristic geometric shapes of this building (1) (Figures 1 and 2).

Several relevant literature references have tried to describe this dome from a geometric and architectural point of view. Some of these references do not specify the surface which best fits the dome, while others claim that this surface is an elliptical paraboloid, though they do not provide any geometric arguments which would substantiate that claim (1, 2, 3, 4, 5, 6).

Perhaps the most important literature references are those from the scientific journal *Informes de la Construcción* (2, 3, 4). In 1990 this journal released a monographic issue on the constructive aspects of Gaudí's work. All the papers were written under the guidance of the Servicio del Patrimonio Arquitectónico de la Diputación de Barcelona (Architectural Heritage Service of the Barcelona Provincial Council, the body owning Palau Güell). These papers analysed several buildings which were being restored at the time or had already undergone a restoration process, such as Palau Güell. The results of those restorations were collected into a set of scientific investigations in that monographic issue. More specifically, three of the nine papers included in issue number 408 of the journal *Informes de la Construcción* focused on analysing several parts of Palau Güell, and all of them described the dome in a brief manner and without providing any geometric justification.

The authors of the abovementioned papers make different claims regarding the geometric type of this dome, but those claims are not supported by any physical or geometric reasoning. On page 19 of the paper by Antoni González and Pablo Carbó (3), the dome is simply referred to as "a classical dome". On page 27 of the paper by Carlos Buxadé and Joan Margarit (4), we find mere observations about the use of paraboloids on the roofing, but it is unclear whether the authors mean the part of the dome projecting from the building or any other architectural element of the roofing. Lastly, the paper by Josep Maria Moreno Lucas (5) does include some deeper insights about the geometric type of the dome, but the claims included are not supported by a rigorous geometric analysis. More precisely, on page 43 it says: "The dome consists of four superimposed concentric rings which define a staggered profile on the outside and a paraboloid of revolution on the inside. The bottom ring rests on a circumference made up by the union of four parabolic arches by means of pendentives."



Figure 1. Photograph showing the dome of the Central Hall, which is the study object of this paper. [Photograph taken by the authors].



Figure 2. Cross-section of the Central Hall in Palau Güell. The dome is highlighted in blue. [This graphic document is reproduced with kind permission of the Arxiu del Servei de Patrimoni Arquitectònic Local de la Diputació de Barcelona].

In addition to the abovementioned monographic issue, among all publications reviewed by us we wish to highlight two books.

First, a book by Daniel Giralt-Miracle (6) which analyses the space, geometry, structure and construction of the most emblematic buildings by Gaudí. On page 41 of this book, in a section entitled "Hyperbolic Paraboloids", it reads as follows regarding the dome: "The *paraboloid of revolution* in Palau Güell, displaying hexagonal decorations and zenithal openings inspired by the Alhambra in Granada, [...]." Besides, on page 136 there is a description of Palau Güell saying: "Inside, the stairway goes over the different levels and reaches the central hall, topped by a *parabolic dome* which passes through the entire building and projects from the rooftop with a conical shape." Second, a more comprehensive book about the history of Palau Güell, published by the Barcelona Provincial Council itself (1). On page 191 of this book, in a section entitled "The sidereal dome" concerning this architectural element, it says: "A canonical dome made up by a *parabola of revolution*, the inside being a pendentive dome or sail dome. From the floor, it seems very light, like a handkerchief held by the corners and blown on the center."

Having examined the most relevant literature references (1, 2, 3, 4, 5, 6, 7), we note that all of them include claims and descriptions with very little geometric data and without any mathematical endorsement. Besides, there is no original drawing or text from Gaudí stating the geometric surface on which this dome is based.

This paper intends to determine the geometry of this dome with a dual purpose:

- To show that the the surface which best fits the dome is not a paraboloid and not even a surface of revolution, thus correcting the widespread misconception about its geometric type.
- 2. To provide the equation and all the geometric parameters of the surface which best fits the dome designed by Gaudí. This information may be useful for several reasons, including:
 - 2.1. In the field of architectural renovation, it is important to have an accurate knowledge of the object to be restored. Having geometric control of the dome would enable the technician to anticipate possible problems which might arise during reconstruction, or even to obtain an accurate estimate of the material costs and the amount of time needed for repairs. The fact that Palau Güell has been listed as World Heritage by the UNESCO makes it important to know the precise geometry of each element, should an intervention become necessary in the future
 - 2.2. In addition to the relevance in the field of architecture and architectural heritage, knowing the geometric parameters and the analytical equation of the dome can help to study it from other points of view. For instance, this equation could help monitor mechanical behaviour and come up with structural hypotheses; it could even help conduct a more detailed analysis of the acoustic impact on the space covered by the dome. It is well known that parabolic and elliptical shapes direct the incident sound waves towards their respective focal points; therefore, these shapes affect the acoustic comfort in spaces (8). Determining the dome's geometric parameters could help to globally control the acoustics and make specialized improvements.

In the next section, we briefly describe the geometric method used to determine the quadric surface which best fits the dome in Palau Güell by Antoni Gaudí. This objective method provides a specific measure of that fit and does not involve mechanical, constructive or structural processes; it only involves standard geometric processes, numerical processes, computing, statistics and 3D data acquisition. We also use this method to find the best-fitting paraboloid. Lastly, using these techniques, we show the geometric parameters of the best-fitting paraboloid and the best-fitting quadric surface. We use the same methodology showed in (9) and (10) for the case of architectural fit by conic regression curves, and in (11) for the case of architectural fit by quadratic regression surface.

2. METHOD TO OBTAIN THE SURFACE WHICH BEST FITS THE DOME AND ITS GEOMETRIC PARAMETERS

2.1. Quadratic surface regression

Let $\mathcal{N} = \left\{ P_i \right\}_{i=1}^{i=n}$ be the point cloud representing the physical surface of the dome which tops the Central Hall in Palau Güell. These points were obtained using photogrammetrical techniques and the software PhotoScan with n = 2154493 (Figure 3). For these points, we use 3D coordinates (x', y', z') according to the 3D orthonormal coordinate system $\mathcal{R}' = \left\{ O; \vec{u}_1, \vec{u}_2, \vec{u}_3 \right\}$ of the scanning device (Figures 4 and 5).

We must point out that the spatial position of this reference system \mathcal{R} ' is unknown before initiating the calculations.

We calculate Γ , which is the regression quadatric surface for \mathcal{N} , and we obtain its general equation, Equation [1], in the reference system \mathcal{R} :

[1]
$$\Gamma \equiv \mathbf{0} = B_{o} x'^{2} + C_{o} y'^{2} + D_{o} z'^{2} + E_{o} x' y' + F_{o} x' z' + G_{o} y' z' + H_{o} x' + I_{o} y' + J_{o} z' + 1$$

This regression surface Γ , the equation of which is Equation [1] in the reference system \mathcal{R}' , is the one which best fits the point cloud $\mathcal{N} = \left\{ P_i = \left(x'_i, y'_i, z'_i \right) \right\}_{i=1}^{i=n}$, minimizing the sum of the quadratic residues $\sum_{i=1}^{i=n} \varepsilon_i^2 = \sum_{i=1}^{i=n} \left(B_o x'_i^2 + C_o y'_i^2 + D_o z'_i^2 + E_o x'_i y'_i + F_o x'_i z'_i + G_o y'_i z'_i + H_o x'_i + I_o y'_i + J_o z'_i + 1 \right)^2$. Matrix equation [2] below derives from the Gauss normal equations which provide the solution to the calculation problem of Γ . These equations have a range of variation 1÷n in Einstein summation convention, being $\mathbf{1}_i = \mathbf{1}$. For example: $x'_i^2 y'_i = \sum_{i=1}^{i=n} x'_i^2 y'_i$, and $\mathbf{1}_i x'_i^3 = \sum_{i=1}^{i=n} x'_i^3$, [2].



The solution of Equation [1] is shown in Section 3 below. After making the classical algebraic calculations for quadratic surface classification, we find that surface Γ is an ellipsoid, because being [3]

$$A = \begin{pmatrix} 1 & \frac{H_0}{2} & \frac{I_0}{2} & \frac{J_0}{2} \\ \frac{H_0}{2} & B_0 & \frac{E_0}{2} & \frac{F_0}{2} \\ \frac{I_0}{2} & \frac{E_0}{2} & C_0 & \frac{G_0}{2} \\ \frac{J_0}{2} & \frac{F_0}{2} & \frac{G_0}{2} & D_0 \end{pmatrix}, A_{00} = \begin{pmatrix} B_0 & \frac{E_0}{2} & \frac{F_0}{2} \\ \frac{E_0}{2} & C_0 & \frac{G_0}{2} \\ \frac{F_0}{2} & \frac{G_0}{2} & D_0 \end{pmatrix}, U = \\ \begin{pmatrix} B_0 & \frac{E_0}{2} & \frac{F_0}{2} \\ \frac{E_0}{2} & C_0 \\ \frac{E_0}{2} & C_0 \\ \frac{E_0}{2} & C_0 \\ \frac{E_0}{2} & D_0 \\ \frac{E$$

we find that detA < 0, $detA_{00} < 0$, $trA_{00} < 0$ and U > 0, and therefore Γ is an ellipsoid.

Next we calculate the orthonormal reference system $\mathcal{R} = \left\{ \theta, \vec{e}_1, \vec{e}_2, \vec{e}_3 \right\}$, where θ is the center of the ellipsoid Γ and $\left\{ \vec{e}_1, \vec{e}_2, \vec{e}_3 \right\}$ are three orthonormal direction vectors for the three axis of Γ , such that \vec{e}_3 is pointed vertically up to the dome. The points in reference \mathcal{R} have coordinates (x, y, z). We note that $\left\{ \vec{e}_1, \vec{e}_2, \vec{e}_3 \right\}$ are eigenvectors of the matrix A_{oo} ,

and the coordinates of $\boldsymbol{\theta}$ are obtained as a solution of system

$$A_{\rm oo}\vec{x}^t = \left(\frac{H_{\rm o}}{2}, \frac{I_{\rm o}}{2}, \frac{J_{\rm o}}{2}\right)^t.$$

The coordinates of the points from cloud \mathcal{N} are changed from reference system \mathcal{R} ' to reference system \mathcal{R} . In addition to this change, we carry out a normalization consisting of the following three steps. First, the entire cloud \mathcal{N} is translated in the direction of vector \vec{e}_3 until point D, the lowest point of \mathcal{N} , is in plane z = 0. Second, we carry out a homothetic transformation of the entire cloud \mathcal{N} with center on θ and homothetic ratio ρ such that the distance between point D and θ is 1. Third, we rotate the entire cloud \mathcal{N} around the axis of \vec{e}_3 until the coordinates of point D in the system \mathcal{R} are (1,0,0). Thus, we obtain a normalized cloud $\mathcal{N} = \left\{ P_i = (x_i, y_i, z_i) \right\}_{i=1}^{i=n}$

with D = (1,0,0) in system $\mathcal{R} = \left\{ \boldsymbol{\theta}; \, \vec{e}_1, \vec{e}_2, \vec{e}_3 \right\}$, where \vec{e}_3 is the direction vector for the transverse axis of cloud \mathcal{N} ; the new coordinates for the points of cloud \mathcal{N} in system \mathcal{R} are (x_{ij}, y_{ij}, z_i) . After all the above calculations, we obtain the new general Equation [4] for Γ , which is the normalized general equation of Γ in system \mathcal{R} :

[4]
$$\Gamma = 0 = B_1 x^2 + C_1 y^2 + D_1 z^2 + E_1 xy + J_1 z + 1$$

In addition to Equation [1] for Γ , Equation [4] provides an easier way to prove that Γ (the quadratic surface which best fits the dome) is an ellipsoid. Later on we will show the geometric parameters of this ellipsoid Γ . In Figure 4 below, the point cloud \mathcal{N} is highlighted in grey colour and the surface Γ is highlighted in red colour. The axis defined by \vec{e}_3 is axis z.



Figure 3. Three-dimensional mesh and textured model based on the point cloud imported into software Photoscan. [Image generated by the authors].



Figure 4. Three-dimensional model of the dome topping the Central Hall in Palau Güell. The cloud \mathcal{N} made up by n = 2154493 points is shown in grey colour. The ellipsoid Γ is shown in red colour. [Image generated by the authors].

2.2. Statistical fit measure

Next, we will calculate to what extent this ellipsoid Γ statistically accounts for point cloud \mathcal{N} . For these calculations, we will use the correlation ratio η^2 , see Equation [5]:

[5]
$$\eta^{2} = 1 - \frac{\sum_{i=1}^{i=n} (z_{i} - f(x_{i}, y_{i}))^{2}}{\sum_{i=1}^{i=n} (z_{i} - \overline{Y})^{2}}$$

where $\overline{Y} = \frac{1}{n} \sum_{i=1}^{i=n} z_i$, and where $(x_i, y_i, f(x_i, y_i))$ are the coordinates of the points forming the regression surface Γ ; or, analitically, where [6]

$$[6] = \frac{-1}{2D_1} \left(J_1 + \sqrt{J_1^2 - 4D_1 - 4B_1D_1x_i^2 - 4C_1D_1y_i^2 - 4D_1E_1x_iy_i} \right)$$

The adjusted correlation ratio η_{adj}^2 is given by Equation [7]:

[7]
$$\eta_{adj}^2 = 1 - \left[(1 - \eta^2) \right] \frac{n - 1}{n - 9 - 1}$$

We know that $\eta_{adj}^2 \in [0,1]$, and the value $\eta_{adj}^2 * 100 = d_{\Gamma}$ is the extent to which the variables $\{Z_i\}_{i=1}^{i=n}$ of cloud $\mathcal{N} = \{P_i = (x_i, y_i, z_i)\}_{i=1}^{i=n}$ are statistically explained by the least squares correlation (Equation [2]) between $\{Z_i\}_{i=1}^{i=n}$ and $\{(x_i, y_i)\}_{i=1}^{i=n}$. In other words, this value d_{Γ} is the percentage by which the variables $\{Z_i\}_{i=1}^{i=n}$ of the points forming cloud \mathcal{N} are statistically explained by the variables z of the points forming the ellipsoid \mathcal{N} . Namely d_{Γ} is a statistical measure of how well the regression ellipsoid Γ fits cloud \mathcal{N} . As already stated, this ellipsoid is the quadric surface which best fits the point cloud \mathcal{N} , and its normalized general equation in the reference system \mathcal{R} is Equation [4].

2.3. Elliptical paraboloid regression

In the process described above, \vec{e}_3 is the direction vector for the geometric axis of the dome in the reference system $\mathcal{R} = \{\theta; \vec{e}_1, \vec{e}_2, \vec{e}_3\}$. Now we calculate the equation of the elliptical paraboloid Δ which best fits the normalized cloud $\mathcal{N} = \{P_i = (x_i, y_i, z_i)\}_{i=1}^{i=n}$. The result is the following normalized general equation, Equation [8], in reference system \mathcal{R} ':

[8]
$$\Delta = 0 = B_2 x^2 + C_2 y^2 + E_2 xy + J_2 z + 1$$

This regression surface Δ , the equation of which is Equation [8] in the reference system \mathcal{R} , is the one which best fits the point cloud $\mathcal{R} = \left\{ P_i = (x_i, y_i, z_i) \right\}_{i=1}^{i=n}$, minimizing the sum of the quadratic residues $\sum_{i=1}^{i=n} \varepsilon_i^2 = \sum_{i=1}^{i=n} (B_2 x_i^2 + C_2 y_i^2 + E_2 x_i y_i + J_2 z_i + 1)^2$. The matrix equation [9] below derives from the Gauss normal equations which provide the solution to the calculation problem of Δ . These equations have a range of variation $1 \div n$ in Einstein summation convention, being $\mathbf{1}_i = \mathbf{1}$.

$$\begin{bmatrix} 9 \end{bmatrix} \begin{pmatrix} \mathbf{1}_{i} x_{i}^{4} & x_{i}^{2} y_{i}^{2} & x_{i}^{3} y_{i} & x_{i}^{2} z_{i} \\ x_{i}^{2} y_{i}^{2} & \mathbf{1}_{i} y_{i}^{4} & x_{i} y_{i}^{3} & y_{i}^{2} z_{i} \\ x_{i}^{3} y_{i} & x_{i} y_{i}^{3} & x_{i}^{2} y_{i}^{2} & x_{i} y_{i} z_{i} \\ x_{i}^{2} z_{i} & y_{i}^{2} z_{i} & x_{i} y_{i} z_{i} & \mathbf{1}_{i} z_{i}^{2} \end{pmatrix} \begin{pmatrix} B \\ C \\ E \\ J \end{pmatrix} = \begin{pmatrix} -\mathbf{1}_{i} x_{i}^{2} \\ -\mathbf{1}_{i} y_{i}^{2} \\ -\mathbf{x}_{i} y_{i} \\ -\mathbf{1}_{i} z_{i} \end{pmatrix}$$

In Figure 5 below, the cloud \mathcal{N} is highlighted in grey colour and the surface Δ is highlighted in light blue colour. The axis defined by \vec{e}_3 is axis z.



Figure 5. Three-dimensional model of the dome topping the Central Hall in Palau Güell. The cloud \mathcal{N} made up of n = 2154493 points is shown in grey colour. The elliptic paraboloid Δ is shown in light blue colour. [Image generated by the authors].

2.4. Statistical fit measure

In Next, we will calculate to what extent this elliptical paraboloid Δ statistically accounts for point cloud \mathcal{N} . For these calculations, we will use the correlation ratio η^2 , see Equation [10]:

[10]
$$\eta^{2} = 1 - \frac{\sum_{i=1}^{i=n} (z_{i} - g(x_{i}, y_{i}))^{2}}{\sum_{i=1}^{i=n} (z_{i} - \overline{Y})^{2}}$$

where $\overline{Y} = \frac{1}{n} \sum_{i=1}^{i=n} z_i$, and where $(x_i, y_i, g(x_i, y_i))$ are the coordinates of the points forming the elliptical paraboloid Δ ; or, analitically, where [11]

[11]
$$g(x_i, y_i) = \frac{-1}{J_2} (B_2 x_i^2 + C_2 y_i^2 + D_2 x_i y_i + 1)$$

The adjusted correlation ratio η_{adj}^2 is given by Equation [12]:

[12]
$$\eta_{adj}^2 = 1 - \left[\left(1 - \eta^2 \right) \right] \frac{n-1}{n-4-1}$$

We know that $\eta_{adj}^2 \in [0,1]$, and the value $\eta_{adj}^2 * 100 = d_{\Delta}$ is the extent to which the variables $\{Z_i\}_{i=1}^{i=n}$ of cloud $\mathcal{N} = \{P_i = (x_i, y_i, z_i)\}_{i=1}^{i=n}$ are statistically explained by the least squares correlation (Equation [9]) between $\{Z_i\}_{i=1}^{i=n}$ and $\{(x_i, y_i)\}_{i=1}^{i=n}$. In other words, this value d_{Δ} is the percentage by which the variables $\{Z_i\}_{i=1}^{i=n}$ of the points forming cloud \mathcal{N} are statistically explained by the variables z of the points forming the elliptical paraboloid Δ . Namely, d_{Δ} is a statistical measure of how well the elliptic paraboloid Δ fits \mathcal{N} . As already stated, this paraboloid is the elliptic paraboloid which best fits the point cloud \mathcal{N} , and its normalized general equation in the reference system \mathcal{R} is Equation [8].

3. NUMERICAL RESULTS

The results of our calculations are displayed graphically in Figures 6 and 7. Nonetheless, in order to complete our paper we also show the numerical results in this section.

After applying the method explained in subsection 2.1 above (Matrix Equation [2], Gauss normal equations), we find that, on the basis of point cloud $\mathcal{N} = \left\{ P_i = (x'_i, y'_i, z'_i) \right\}_{i=1}^{i=n}$ in $\mathcal{R}' = \left\{ O; \vec{u}_1, \vec{u}_2, \vec{u}_3 \right\}$, Equation [1] of the ellipsoid Γ has the following coefficient values: $B_0 = -0.0033$, $C_0 = -0.0028$, $D_0 = -0.0012$, $E_0 = -0.0008$, $F_0 = -0.0013$, $G_0 = -0.0027$, $H_0 = -0.0531$, $J_0 = -0.0378$.

[1]
$$\Gamma \equiv 0 = B_o x'^2 + C_o y'^2 + D_o z'^2 + E_o x' y' + F_o x' z' + G_o y' z' + H_o x' + I_o y' + J_o z' + 1$$

Using C++ language, we ourselves have created all the computer programs needed for the numerical analysis methods used for calculation in this paper.

By means of the geometric transformations explained in the previous section, we find that the normalized general equation [4] of the ellipsoid Γ in the reference system \mathcal{R} has the following coefficient values: $B_1 = -0.9456$, $C_1 = -0.9362$, $D_1 = -0.0759$, $E_1 = -0.0087$, $J_0 = -0.3181$.

[4]
$$\Gamma = 0 = B_1 x^2 + C_1 y^2 + D_1 z^2 + E_1 xy + J_1 z + 1$$

After applying the method explained in subsection 2.3 above (Matrix Equation [9], Gauss normal equations), we find that the normalized general equation [8] of the elliptical paraboloid Δ in the reference system \mathcal{R} has the following coefficient values: $B_2 = -0.8640$, $C_2 = -0.8526$, $E_2 = -0.0022$, $J_2 = -0.4577$.

8]
$$\Delta = 0 = B_2 x^2 + C_2 y^2 + E_2 x y + J_2 z + 1$$



Figure 6. The n = 2154493 points of cloud \mathcal{N} making up the dome are shown in grey colour. The ellipsoid \mathcal{N} is shown in red colour. [Image generated by the authors].



Figure 7. The n = 2154493 points of cloud \mathcal{N} making up the dome are shown in grey colour. The elliptical paraboloid Δ is shown in light blue colour. [Image generated by the authors]

Using these values in the equations [5-7] and [10-12], we find that the quadric surface which best fits point cloud \mathcal{N} is the ellipsoid Γ , and the statistical measure of that fit is $d_{\Gamma} = 99.79 \%$. Similarly, the elliptical paraboloid which best fits \mathcal{N} is paraboloid Δ , and the statistical measure of that fit is $d_{\Delta} = 99.19 \%$. This difference in fit measure can be visualized in Figures 6 and 7.

4. GEOMETRIC PARAMETERS OF ELLIPSOID Γ

As stated before, we start from the reference system $\mathcal{R}' = \left\{ O; \vec{u}_1, \vec{u}_2, \vec{u}_3 \right\}$. All calculations and all coordinates mentioned in this paper are based on this reference system. Using system \mathcal{R}' , we find the new reference system $\mathcal{R} = \left\{ \theta; \vec{e}_1, \vec{e}_2, \vec{e}_3 \right\}$, where $\left\{ \vec{e}_1, \vec{e}_2, \vec{e}_3 \right\}$ are eigenvectors of the matrix A_{oo} . The coordinates of θ are obtained as a solution of system $A_{oo}\vec{x}^t = \left(\frac{H_o}{2}, \frac{I_o}{2}, \frac{J_o}{2} \right)^t$. Thus, θ is the center of the ellipsoid Γ , and $\left\{ \vec{e}_1, \vec{e}_2, \vec{e}_3 \right\}$ are three orthonormal direction vectors for the three axis of Γ , such that \vec{e}_3 is pointed vertically up to the dome. We change the coordinates of the points forming the cloud \mathcal{N} in such a way that the lowest point has coordinates (1,0,0) in the system \mathcal{R} , and thus we obtain the normalised general equation [4] of ellipsoid Γ in the system \mathcal{R} . This ellipsoid Γ is the quadric surface which best fits the dome.

The reader may now repeat our calculations and check that, in the system \mathcal{R} , the center θ of the ellipsoid Γ has coordinates $\theta = (0, 0, -2.0966)$. The vertices C, C' of the major axis of Γ have coordinates C = (0, 0, 2.0962) and C' = (0,0, -6.2894). Parameter c of the semi-major axis is $c = d(\theta, C) = 4.1928$. The vertices A, A' and B, B' (these vertices are on the plane which meets the center θ and is perpendicular to \overline{e}_3) have coordinates A = (0.4333, 1.1045, -2.0966), A' = (0.4333, -1.1045, -2.0966)

and B = (1.112, -0.4363, -2.0966), B' = (-1.112, 0.4363, -2.0966). Parameters a and b of the semi-minor axes are $a = d(\theta, A) =$ 1.1864 and $b = d(\theta, B) =$ 1.1946.

Table 1 summarizes the numerical values of the geometric parameters of Γ in the reference system \mathcal{R} .

Table 1. Geometric parameters of Γ in the reference system \mathcal{R} .

Point	Coordinates	Point	Coordinates
vertex C	(0, 0, 2.10)	center θ	(0,0, -2.10)
vertex C'	(0,0, -6.29)	focal F_{ac}^{+}	(0,0,1.92)
vertex A	(0.43,1.10, -2.10)	focal F_{ac}^-	(0,0, -6.12)
vertex A'	(-0.43, -1.10, -2.10)	focal F_{bc}^+	(0,0,1.92)
vertex B	(1.11, -0.44, -2.10)	focal F_{bc}^{-}	(0,0, 6.12)
vertex B'	(-1.11,0.44, -2.10)		
Parameter	Value	Parameter	Value
$d(\theta, C)$	4.19	ϵ_{ac}	0.959
$d(\theta, A)$	1.186	ϵ_{bc}	0.958
$d(\theta, B)$	1.194		

With all the above, we can consider *S*, which is the reduced orthonormal reference system of ellipsoid Γ where $\vec{v}_1 = \frac{\vec{\Theta A}}{\|\vec{\Theta A}\|}$, $\vec{v}_2 = \frac{\vec{\Theta B}}{\|\vec{\Theta B}\|}$ and $\vec{v}_3 = \frac{\vec{\Theta C}}{\|\vec{\Theta C}\|}$; and then we obtain the canonical equation of Γ on system *S*. If points have coordinates $(\bar{x}, \bar{y}, \bar{z})$ in this system *S*, then the canonical equation of Γ is as follows [13]:

[13]
$$\Gamma \equiv 1 = \frac{\overline{x}^2}{1.1864^2} + \frac{\overline{y}^2}{1.1946^2} + \frac{\overline{z}^2}{4.1928^2}$$

But the canonical equation [13] is not the intrinsic equation of ellipsoid Γ , because the length unit has been determined in a subjective way (we have considered that the distance from the axis of \vec{e}_{a} to point *D* is 1). Therefore, we finally con-

sider
$$E = \{\theta; \vec{w}_1, \vec{w}_2, \vec{w}_3\}$$
, being $\vec{w}_1 = \overrightarrow{\theta A}, \vec{w}_2 = \overrightarrow{\theta B} \frac{\|\theta A\|}{\|\overline{\theta B}\|}$ and

 $\vec{w}_{3} = \overrightarrow{\Theta C} \frac{\|\Theta A\|}{\|\Theta C\|}$. This reference system *E* is indeed the intrinsic reference system of the ellipsoid Γ , because as the length unit

we take parameter $a = d(\theta, A) = \|\overline{\theta A}\|$, which is the smallest of the three distances from the ellipsoid's vertices to its center θ . Then, since a = 1 and all points have coordinates (x, y, z)in this system *E*, the intrinsic canonical equation [14] of Γ is as follows:

[14]
$$\Gamma = 1 = \dot{x}^2 + \frac{\dot{y}^2}{1.0069^2} + \frac{\dot{z}^2}{3.5341^2}$$

By means of calculation, we find that, in order to position the reference system \mathcal{R} on the reference system S we need a rotation of 21.4214° degrees.

The intersection of the ellipsoid Γ to the plane of equation $\overline{y} = \mathbf{0}$ in the system *S* (that is, the plane containing the center θ and the vertices *A*, *C*) is an ellipse Σ_{ac} . The eccentricity of this ellipse is $\varepsilon_{ac} = 0.9591$. The intersection of the ellipsoid Γ to the plane of equation $\overline{x} = \mathbf{0}$ in the system *S* (that is, the plane containing the center θ and the vertices *B*, *C*) is an ellipse Σ_{bc} . The eccentricity of this ellipse is $\varepsilon_{bc} = 0.9585$.

The focal points F_{ac}^{+} , F_{ac}^{-} of the ellipse Σ_{ac} are located on the axis of the direction vector $\vec{e}_{_3}$, and their coordinates on the system \mathcal{R} are $F_{ac}^{+} = (0,0,1.9224)$, $F_{ac}^{-} = (0,0,-6.1156)$. The focal points F_{bc}^{+} , F_{bc}^{-} of the ellipse Σ_{bc} are located on the axis of the direction vector $\vec{e}_{_3}$, and their coordinates on the system \mathcal{R} are $F_{ac}^{+} = (0,0,1.9249)$, $F_{ac}^{-} = (0,0,-6.1180)$, see Table 1.

As for the geometric parameter related to surface fracture in case of deformation (that is, the Gaussian curvature) and the geometric parameter related to the minimum possible surface area (that is, the mean curvature), Figure 8 shows both parameters with a colour gradation. The vertical axis in this Figure is determined by \vec{e}_{a} .

5. GEOMETRIC ANALYSIS OF ALSINA'S CROSS SECTION

As already mentioned in the introduction, there is no known original document from Gaudí explaining how this dome was designed or built, and there are no quotes from Gaudí describing its geometric type. There is, however, a graphic document of that period which includes this architectural element. It is a cross section of the Central Hall in Palau Güell (1). It was drawn by Joan Alsina i Arús, at the request of Eusebi Güell, for an exhibition dedicated to Gaudí that was held at the Grand-Palais of Paris in 1910 (Figure 9). Joan Alsina i Arús was professor of descriptive geometry at the Escuela Técnica Superior de Arquitectura de Barcelona at the time, suggesting that this graphic document is geometrically accurate.

Therefore, we believe it is important to incorporate into our paper a geometrical analysis of the dome appearing in the cross section drawn by Joan Alsina i Arús. This analysis will enable us to identify the type of arc used to represent the dome, and thus we will infer the type of surface that Joan Alsina i Arús wanted to depict on the document. For this geometrical analysis we have used a method which is described in full detail in (9). A summary description can also be found in (10). Nonetheless, the steps involved in this method are briefly outlined below: We start from the point cloud \mathcal{M} outlining the arc (in this case, it is made up of 45 points, see Figure 9); then we use the Gauss normal equations to calculate the regression conical curve Q which best fits the cloud \mathcal{M} ; then we change from the initial reference system to reference system \mathcal{T} , consisting of the center and the axes of Q; then, in system \mathcal{T} , we find the equations of the five regression curves (ellipse, parabola, hyperbola, catenary and Rankine) which best fit the cloud \mathcal{M} , solving the corresponding Gauss equations; and finally we determine the statistical fit measure for each curve. We insist that the reader may turn to (10) and, specially, (9) for a more detailed explanation of this method. In order no to make this paper unnecessarily long, the results are displayed in graphical form only. Figure 9 shows the ellipse and the parabola which best fit the point cloud \mathcal{M} taken from the arc drawn by Alsina i Arús. The rest of regression curves (hyperbola, catenary, Rankine) are not included because they have a worse fit and they add nothing to this paper.



Figure 8. The image on the right shows the Gaussian curvature with a colour gradation. The image on the left shows the mean curvature with a colour gradation. [Images generated by the authors.]



Figure 9. On the left, cross section of Palau Güell as drawn by Joan Alsina i Arús in 1910 (1, pp.42). Top right: set \mathcal{M} of 45 points outilining the arc drawn by Joan Alsina i Arús (we have also highlighted this arc in orange colour on the original document). Center right: regression ellipse in red colour. Bottom right: regression parabola in blue colour.

Thus, despite all the literature consulted by us claims that the dome in Palau Güell corresponds to a paraboloid, the geometric analysis of the arc drawn by Alsina i Arús shows that the conical curve which best fits the dome's cross section is an ellipse, and the statistical measure of that fit is 99.85%.

6. CONCLUSIONS

After the calculations in sections 3 and 4, we have ascertained that, contrary to what is claimed in the specialised literature, the surface which best fits the dome in Palau Güell is an ellipsoid, and Joan Alsina i Arús also expressed this graphically. In mathematical terms: In view of the statistical measure $d_{\Gamma} = 99.79\%$ for the fit of the ellipsoid Γ (best-fitting quadric surface), we can claim there is sufficient statistical evidence that the dome was designed based on the geometry of an ellipsoid. This fit can be visualized in Figure 6. We have also calculated and provided the equations of the elliptical paraboloid Δ (best-fitting paraboloid) as well as the measure $d_{\Delta} = 99.19\%$ of its fit. This fit is substantially lower and does not provide sufficient statistical evidence that the dome was designed based on the geometry of a paraboloid. The reader may visually perceive this difference in fit through Figures 6 and 7.

As already stated, it is commonly claimed that the dome in Palau Güell is a paraboloid and, specifically, a paraboloid of revolution. However, we have proved that the ellipsoid Γ is not an ellipsoid of revolution. More specifically:

In the intrinsic canonical equation [14]

[14]
$$\Gamma = 1 = \dot{x}^2 + \frac{\dot{y}^2}{1.0069^2} + \frac{\dot{z}^2}{3.5341^2}$$

we see that the difference in length between the two semiminor axes is 1.0069 - 1 = 0.0069 length units. If Γ were to be an ellipsoid of revolution, this difference would have to be 0. The length unit of the real dome is more or less 3 meters, and the difference in length between the two real semi-minor axes is 0.0069 * 3 = 0.0207 m. Therefore, the difference between the two minor diameters is 0.0207 * 100 * 2 = 4.14 cm. Admittedly, this difference should be 0.00 cm if this was a dome of revolution. Nonetheless, such a minor discrepancy is not material enough to assess if the dome was or was not intentionally designed as a dome of revolution. This difference in diameter could be interpreted as a construction error, or even as mechanical settlements of the dome.

Despite there is no single document indicating whether Gaudí intentionally designed a surface a revolution or not, based on our investigation we can claim that quadric surface which best fits the dome of the Central Hall in Palau Güell is an ellipsoid, and not a paraboloid.

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